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# MARANGONI INSTABILITY IN CASE OF A NON-UNIFORM HEAT SOURCE

Y. T. Shah and Andras Szeri
Department of Chemical and Petroleum Engineering, University of Pittsburgh, Pittsburgh, Pennsylvania 15261, U.S.A.

and

Department of Mechanical Engineering, University of Pittsburgh, Pittsburgh, Pennsylvania 15261, U.S.A.

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## NOMENCLATURE

| | |
|---|---|
| $B$, | Marangoni number $\dfrac{\sigma(T_1-T_2)d}{\rho\nu\bar{\alpha}}$; |
| $d$, | thickness of liquid layer; |
| $f, g_1, g_2$, | variables defined by equations (7)–(10); |
| $h$, | convective heat-transfer coefficient; |
| $K$, | thermal conductivity of liquid; |
| $L$, | Biot number [hd/K]; |
| $R$, | dimensionless heat generation parameter as defined by equations (1) and (2); |
| $t$, | time; |
| $T$, | temperature; |
| $T'$, | perturbation temperature; |
| $v$, | perturbation velocity in $y$-direction; |
| $x, y, z$, | cartesian coordinate system. |

Greek letters

| | |
|---|---|
| $\rho$, | density of the liquid layer; |
| $\bar{\alpha}$, | thermal diffusivity of the liquid layer; |
| $\nu$, | kinematic viscosity of the liquid layer; |
| $\alpha$, | wave number for the periodic fluctuations; |
| $\xi, \eta, \zeta$, | dimensionless $x, y, z$ direction distance variables $(x/d, y/d, z/d)$; |
| $\varepsilon$, | dimensionless ratio of reaction film thickness to the actual thickness of the liquid layer; |
| $\Gamma$, | parameter defined by equation (7); |
| $\sigma$, | the rate of change of surface tension with temperature evaluated at gas–liquid interface temperature; |
| $\Psi_1, \Psi_2$, | variables defined by equations (16) and (17). |

## INTRODUCTION

We consider in this communication the problem of surface tension driven convective instability in a liquid layer in which heat is generated in a thin layer near the gas–liquid interface due to either a radiation catalysed reaction or a zeroth order gas–liquid reaction. This practical case is interesting because it gives curious results for critical Marangoni number for a stationary neutral stability curve. Furthermore, this problem leads to the analysis of surface-tension driven instability in the presence of a type of a non-linear temperature profile that has not been considered as yet.

## THEORY

### *Steady state equations*

We consider a stationary liquid layer infinite in the horizontal $x$ and $z$ directions and in which a vertical distance $y$ is measured from the gas–liquid interface, i.e. $y = 0$ corresponds to the gas–liquid interface and $y = d$ corresponds to the lower solid surface. The temperatures of the lower and upper surfaces are $T_1$ and $T_2$ respectively. Except for the heat generation in the reaction film, heat transfer in the liquid layer takes place by conduction. Under steady state conditions, the temperature distribution in the liquid depends only on $y$ and this distribution can be shown as

$$\frac{T-T_2}{T_1-T_2} = (1-R\varepsilon^2+2R\varepsilon)\eta - R\eta^2 \quad 0 \leqslant \eta \leqslant \varepsilon \tag{1}$$

and

$$\frac{T-T_2}{T_1-T_2} = R\varepsilon^2+(1-R\varepsilon^2)\eta \quad \varepsilon \leqslant \eta \leqslant 1 \tag{2}$$

where $\eta = y/d$, $R = Sd^2/2K(T_1-T_2)$, $\varepsilon$ is the dimensionless reaction film thickness; $\varepsilon = y_r/d$, $y_r$ being the dimensional reaction film thickness. $S$ is the uniform rate of heat generation in the reaction film. Few typical temperature distributions when $R > 0$, are described in Fig. 1.

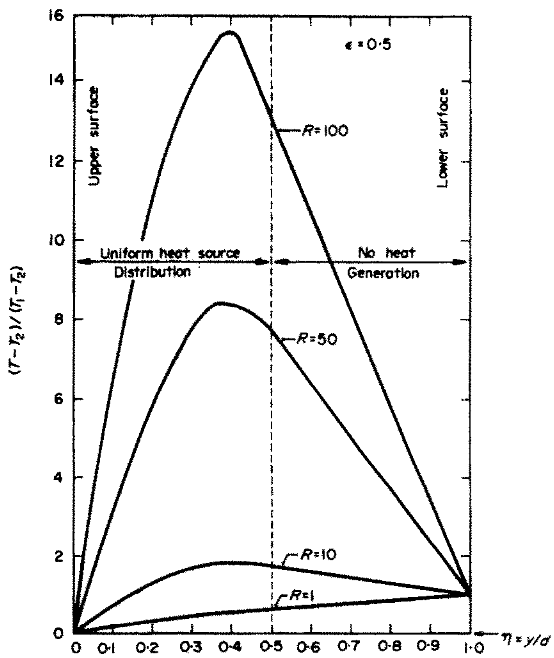


Fig. 1. Steady-state temperature distribution for heat source uniform over $\varepsilon = 0{\cdot}5$. Effect of $R$.

*Small perturbation analysis*

Assuming that only surface-tension and temperature vary within the liquid layer; gas–liquid interface retains its planar state and that surface viscosity and elasticity effects are negligible, the relevant equations for small velocity and temperature perturbations are (1):

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla^2 v = 0 \tag{3}$$

and

$$\left(\frac{\partial}{\partial t}-\bar{\alpha}\nabla^2\right)T' = -v\frac{\partial T}{\partial y} \tag{4}$$

with conditions.

$$v = 0,\quad \rho\nu\frac{\partial^2 v}{\partial y^2} = \sigma\nabla_1^2 T_1';\quad -K\frac{\partial T'}{\partial y} = hT' \ \text{at}\ y = 0 \tag{5}$$

$$v = \frac{\partial v}{\partial y} = 0;\quad T' = \Gamma\frac{\partial T'}{\partial y}\ \text{at}\ y = d \tag{6}$$

where $\Gamma$ is a constant, and

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}.$$

We assume infinitesimal perturbations $v$ and $T'$ to be of the form

$$v = -\frac{\bar{\alpha}}{d}F(\xi,\zeta)f(\eta)\,e^{p\tau} \tag{7}$$

and

$$T' = (T_1-T_2)F(\xi,\zeta)g(\eta)\,e^{p\tau} \tag{8}$$

which gives for stationary marginal state ($p = 0$).

$$(D^2-\alpha^2)^2 f = 0,\quad (D^2-\alpha^2)g_1 = [(1-R\varepsilon^2+2R\varepsilon)-2R\eta]f \quad \text{for } 0<\eta<\varepsilon \tag{9}$$

$$(D^2-\alpha^2)^2 f = 0,\quad (D^2-\alpha^2)g_2 = (1-R\varepsilon^2)f \quad \text{for } \varepsilon<\eta<1 \tag{10}$$

where $\alpha$ is related to $F(\zeta,\xi)$ by the relation

$$\frac{\partial^2 F}{\partial\xi^2}+\frac{\partial^2 F}{\partial\zeta^2}+\alpha^2 F = 0 \tag{11}$$

with conditions

$$f(1) = f'(1) = f(0) = 0\quad g'(0) = Lg(0),\quad f''(0) = B\alpha^2 g(0),$$
either $g(1) = 0$ conducting case (12)
or $g'(1) = 0$ insulating case
and $g_1(\varepsilon) = g_2(\varepsilon)$; $g_1'(\varepsilon) = g_2'(\varepsilon)$.

*Solution*

A solution for $f(\eta)$ subjected to the condition (11) is

$$f(\eta) = C_1\sinh\alpha\eta + C_2\eta\sinh\alpha\eta + C_3\eta\cosh\alpha\eta$$

where

$$\frac{C_2}{C_1} = \frac{(\sinh\alpha)(\cosh\alpha)}{\alpha}-1; \tag{13}$$

$$\frac{C_3}{C_1} = -\frac{(\sinh^2\alpha)}{\alpha}. \tag{14}$$

The solutions for $g_1(\eta)$ and $g_2(\eta)$ can be obtained as

$$g_1(\eta) = B_1\sinh\alpha\eta + B_2\cosh\alpha\eta + \Psi_1(\eta) \tag{15}$$

$$g_2(\eta) = B_3\sinh\alpha\eta + B_4\cosh\alpha\eta + \Psi_2(\eta) \tag{16}$$

where

$$\Psi_1(\eta) = \tfrac{1}{2}\Bigg[\frac{C_1\Lambda_1\eta}{\alpha}\cosh\alpha\eta + (C_2\Lambda_1 - C_1\Lambda_2)\frac{\eta^2}{2\alpha}\cosh\alpha\eta - (C_2\Lambda_1 - C_1\Lambda_2)\frac{\eta}{2\alpha^2}\sinh\alpha\eta + \frac{C_3\Lambda_1}{2\alpha}\eta^2\sinh\alpha\eta - \frac{C_3\Lambda_1}{2\alpha^2}\eta\cosh\alpha\eta - \frac{C_2\Lambda_2}{\alpha}\left(\frac{\eta^3}{3}+\frac{\eta}{2\alpha^2}\right)\cosh\alpha\eta + \frac{C_2\Lambda_2}{2\alpha^2}\eta^2\sinh\alpha\eta - \frac{C_3\Lambda_2}{\alpha}\left(\frac{\eta^3}{3}+\frac{\eta}{2\alpha^2}\right)\sinh\alpha\eta + \frac{C_3\Lambda_2}{2\alpha^2}\eta^2\cosh\alpha\eta\Bigg] \tag{17}$$

$$\Lambda_1 = 1-R\varepsilon^2+2R\varepsilon \tag{18}$$

$$\Lambda_2 = 2R \tag{19}$$

and

$$\Psi_2(\eta) = (1-R\varepsilon^2)\Bigg[\frac{-C_2}{4\alpha^2}\eta\sinh\alpha\eta + \left(\frac{C_1}{2\alpha}-\frac{C_3}{4\alpha^2}\right)\times\eta\cosh\alpha\eta + \left(\frac{C_3}{4\alpha}\right)\eta^2\sinh\alpha\eta + \left(\frac{C_2}{4\alpha}\right)\eta^2\cosh\alpha\eta\Bigg] \tag{20}$$

where the constants $B_1$ to $B_4$ are obtained from conditions (12). The desired solution for Marangoni number $B$ is

$$B = \frac{2C_2}{\alpha B_2}. \tag{21}$$

Above solution gives $B$ as a function of $L$, $R$, $\alpha$ and $\varepsilon$. For a set of values of $L$, $R$ and $\varepsilon$, the minimum in $B$ vs $\alpha$ plot represents the critical value of $B$. We briefly describe below the behavior of this $B_{\text{critical}}$.

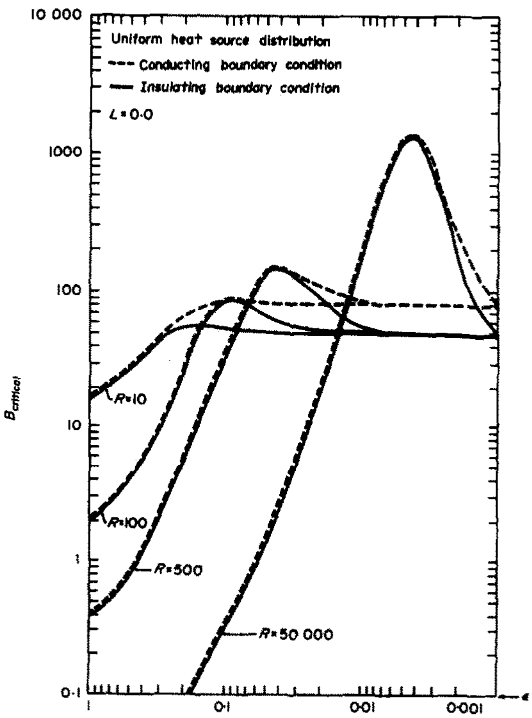


FIG. 2. Variation of critical with $\varepsilon$ at various $R$.

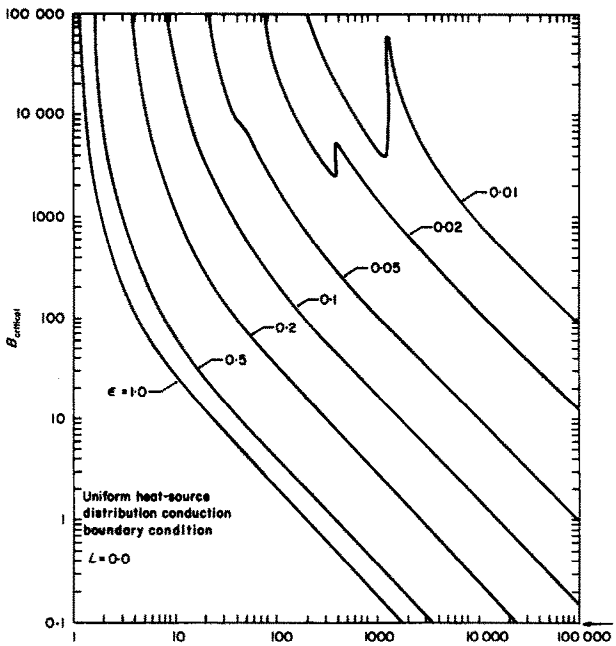


FIG. 3. $B_{critical}$ vs $R$ at various $\varepsilon$.

RESULTS

Consider first the case $R > 0$, $L = 0$. Figure 2 describes the typical effects of $\varepsilon$ and $R$ on $B_{critical}$ for this case. These results appear reasonable in light of the work of Vidal and Acrivos [2] and due to the fact that in the limits $\varepsilon \to 0$, $R$ finite and $R \to 0$ Pearson's [1] results are recovered. Figure 2 shows that for constant $R$, $B_{critical}$ vs $\varepsilon$ plot exhibits a maximum. This maximum is more predominant at higher values of $R$. It is interesting to note that the maximum always occurs at the point where the temperature gradient in the unperturbed state vanishes at the lower plate. The location of the maximum is thus defined by the expression $\varepsilon^{1/2}R = 1$. Similar types of curious results for $B_{critical}$ $(\varepsilon, R)$ were obtained for the case $R < 0$ $(T_1 < T_2, S > 0)$ and $L = 0$. This behavior is illustrated in Fig. 3. Although the results are described for the conducting case, almost identical results were obtained for the insulating case, $B_{critical}$ for this case tends to infinity, as it should, when $|R| \to |1/(\varepsilon^2 - 2\varepsilon)|$. For $|R|$ less than $|1/(\varepsilon^2 - 2\varepsilon)|$, $dT/d\eta$ would be negative at all $\eta$ and hence Marangoni instability is not possible. For small $\varepsilon$, plots of $B_{critical}$ vs $R$ show curious humps. Since at large $|R|$, $dT/d\eta$ in the reaction film is almost independent of sign on $R$, $B_{critical}(\varepsilon)$ shown in Figs. 2 and 3 are almost identical at large $|R|$.

Similar behavior of $B_{critical}(R, \varepsilon)$ were observed for non-zero values of $L$. As $L$ increased, $B_{critical}(R, \varepsilon)$ was however, increased.

The critical wave number ($\alpha_{critical}$) corresponding to $B_{critical}$ increased with a decrease in $\varepsilon$ at constant $R$.

REFERENCES

1. J. R. A. Pearson, On convection cells induced by surface tension, *J. Fluid Mech.* 4 (5), 489–500 (1958).
2. A. Vidal and A. Acrivos, Effect of non-linear temperature profiles on the onset of convection driven by surface tension gradients, *IEC Fundamentals* 7, 53–58 (1968).